
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Assessment Examination

FORM VI

MATHEMATICS EXTENSION 2

Thursday 19th May 2016

General Instructions

- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 73 boys

Examiner

DWH

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The parametric equations of an ellipse are:

(A) $x = a\cos\theta$ and $y = b\tan\theta$

(B) $x = a\cos\theta$ and $y = b\sin\theta$

(C) $x = a\sec\theta$ and $y = b\tan\theta$

(D) $x = a\sec\theta$ and $y = b\sin\theta$

QUESTION TWO

Determine $\displaystyle\int \frac{1}{\sqrt{16-(x+3)^2}}\,dx$.

(A) $\dfrac{1}{4}\sin^{-1}\left(\dfrac{x+3}{4}\right)+C$

(B) $\dfrac{1}{4}\tan^{-1}\left(\dfrac{x+3}{4}\right)+C$

(C) $\sin^{-1}\left(\dfrac{x+3}{4}\right)+C$

(D) $\tan^{-1}\left(\dfrac{x+3}{4}\right)+C$

QUESTION THREE

The number $2(\cos\pi - i\sin\pi)$ is:

(A) rational

(B) undefined

(C) irrational

(D) purely imaginary

Examination continues next page ...

QUESTION FOUR

A polynomial $P(x)$ with real coefficients has odd degree. Which of the following sentences must be correct?

(A) The minimum number of real zeroes of $P(x)$ is 1.

(B) The minimum number of real zeroes of $P(x)$ is 0.

(C) The minimum number of non-real zeroes of $P(x)$ is 1.

(D) The minimum number of real zeroes of the derivative $P'(x)$ is 1.

QUESTION FIVE

The equation $x^4 + 2x^3 + 8x + 16 = 0$ has a double root at:

(A) $x = 1 - \sqrt{3}i$

(B) $x = 1 + \sqrt{3}i$

(C) $x = -2$

(D) $x = 2$

QUESTION SIX

Consider the integral $I = \displaystyle\int_{-2}^{4} x^3\sqrt{16 - x^2}\,dx$. Which is a true statement?

(A) $I = \displaystyle\int_{2}^{4} x^3\sqrt{16 - x^2}\,dx$

(B) $I = 2\displaystyle\int_{0}^{2} x^3\sqrt{16 - x^2}\,dx + \int_{2}^{4} x^3\sqrt{16 - x^2}\,dx$

(C) $I = \displaystyle\int_{-4}^{-2} x^3\sqrt{16 - x^2}\,dx$

(D) $I = \displaystyle\int_{-4}^{-2} x^3\sqrt{16 - x^2}\,dx + 2\int_{-2}^{0} x^3\sqrt{16 - x^2}\,dx$

QUESTION SEVEN

A polynomial $P(x)$ has real coefficients and $P(3i) = 0$. Which of the following must be true?

(A) $P(x)$ has a quadratic factor that has no real roots.

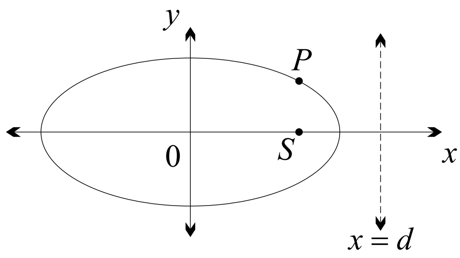
(B) $P(3) = i$.

(C) $P\left((3i)^2\right) = 0$.

(D) $P(x)$ is a polynomial of odd degree.

Examination continues overleaf . . .

QUESTION EIGHT



An ellipse centred at the origin has a focus at $S(3,0)$ and a directrix at $x = d$, where $d > 0$. The ellipse also passes through the point $P\left(3, \frac{16}{5}\right)$.

Find d if the eccentricity of the ellipse is $\frac{3}{5}$.

(A) $d = \frac{123}{25}$

(B) $d = \frac{48}{25}$

(C) $d = \frac{25}{3}$

(D) $d = \frac{13}{3}$

QUESTION NINE

The following graph displays the distance x of a particle from a fixed point O over time t.

Which of the following could NOT describe a possible motion of the particle for $t \geq 0$?

(A) The particle is at rest.

(B) The acceleration of the particle is constant.

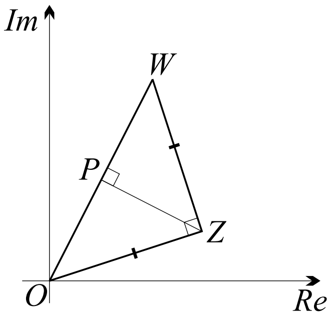
(C) The particle is undergoing uniform circular motion about O.

(D) The path of the particle is a parabola.

Examination continues next page ...

QUESTION TEN

The point Z represents the complex number z. The intervals OZ and ZW are equal in length and perpendicular, as shown. The point P is the foot of the altitude from Z to OW.



The point P represents which complex number?

(A) $\frac{1}{2}iz$

(B) $\frac{1}{2}iz^2$

(C) $\frac{1}{2}(z + iz)$

(D) $\frac{1}{2}(z - iz)$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int \frac{x^2}{\sqrt{1-x^3}}\,dx\,.$ **2**

(b) Express the polynomial $P(x) = x^3 - 5x^2 + 11x - 15$ as a product of linear factors. **3**

(c) (i) Expand $(x-1+i)\,(x-1-i)$. **1**

(ii) Consider the polynomial $P(x) = x^3 + kx - 5$, where k is real. The remainder when $P(x)$ is divided by $x^2 - 2x + 2$ is $9x - 9$. Find k. **2**

(d) (i) Find values of A, B and C, such that **2**

$$\frac{4x^2+11}{(2x+3)(x+4)} = A + \frac{B}{2x+3} + \frac{C}{x+4}\,.$$

(ii) Hence, or otherwise, find $\displaystyle\int_{-1}^{1} \frac{4x^2+11}{(2x+3)(x+4)}\,dx\,.$ **2**

(e) A block of mass 4 kg moves in a straight line across a flat surface. At time t seconds its displacement from a fixed origin is x metres and its velocity is v metres per second. The variable force acting on the block is $18 - 8x$ Newtons. When $x = 4$, $v = 2$. **3**

Find v^2 in terms of x.

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the hyperbola $x = 3\sec\theta$, $y = 2\tan\theta$.

(i) Find the Cartesian equation of the hyperbola. **1**

(ii) Find the equations of the asymptotes of the hyperbola. **1**

(iii) Find the foci of the hyperbola. **2**

(iv) Sketch the hyperbola, showing asymptotes and any intercepts with axes. **2**

(b) An object of mass m is dropped and is then subject to gravity and air resistance. When its displacement is x metres, its velocity is $v\,\text{ms}^{-1}$. The magnitude of forces acting on the object are gravity mg Newtons, and air resistance kv^2 Newtons, for some positive constant k. Take downwards as positive. **2**

Write down an expression for the acceleration of the object and hence find an expression for the terminal velocity V_T.

(c) Use the substitution $x = 2\sin\theta$ to find $\displaystyle\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$. **3**

(d) Consider the polynomial $P(x) = x^3 - 5x^2 - 2x - 8$ with zeroes α, β and γ.

(i) Find a simplified polynomial expression $Q(x)$ with zeroes $\dfrac{\alpha}{2}$, $\dfrac{\beta}{2}$ and $\dfrac{\gamma}{2}$. **2**

(ii) Hence, or otherwise, find a polynomial $T(x)$ with zeroes $\dfrac{3\alpha}{2} + \beta + \gamma$, $\alpha + \dfrac{3\beta}{2} + \gamma$ and $\alpha + \beta + \dfrac{3\gamma}{2}$. **2**

Examination continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) You may assume the equation of the tangent to a hyperbola is $\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$ and the equation of the chord of contact is $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = 1$. Do NOT prove these equations.

For the hyperbola $\dfrac{x^2}{225} - \dfrac{y^2}{144} = 1$, find:

(i) the chord of contact from the point $(15, 6)$, and **1**

(ii) the two tangents to the hyperbola passing through the point $(15, 6)$. **3**

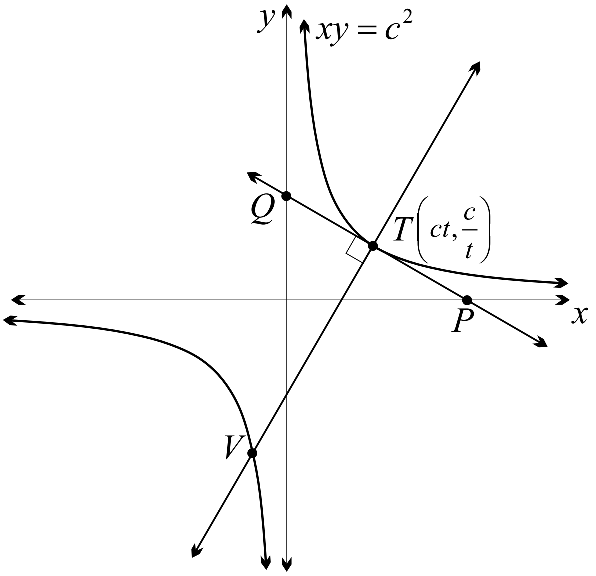
(b) Consider the integral $I_n = \displaystyle\int_0^1 \left(1 + x^2\right)^n dx$, where n is a non-negative integer. Use integration by parts to show that for $n \geq 1$, $I_n = \dfrac{2^n}{2n+1} + \dfrac{2n}{2n+1} I_{n-1}$. **3**

(c) Consider the polynomial equation $x^3 - 3x^2 - 2x - 1 = 0$ with roots α, β and γ.

(i) Find the polynomial equation with roots α^2, β^2 and γ^2. **2**

(ii) Find $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$. **1**

(d) The tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ meets the x-axis at P and the y-axis at Q. The normal at T meets the other arm of the hyperbola at V.



You may use the equation for the tangent at T given by $x + t^2y = 2ct$ and the equation for the normal at T given by $t^2x - y = ct^3 - \dfrac{c}{t}$. Do NOT prove these.

(i) Find the co-ordinates of P and Q. **1**

(ii) Show that the co-ordinates of V are $\left(-\dfrac{c}{t^3}, -ct^3\right)$. **2**

(iii) Show that $\triangle PQV$ is isosceles. **2**

Examination continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Show that the equation of the normal to an ellipse at the point $(a\cos\theta, b\sin\theta)$ is given by $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$. **3**

(b) A ball of unit mass is projected vertically upwards from ground level with initial velocity $U\,\text{ms}^{-1}$.

Take the point of launch as the origin. At t seconds after launch, the displacement from the origin is x metres and the velocity is $v\,\text{ms}^{-1}$. Take upward motion as positive.

The resistive force due to its passage through the air is proportional to the velocity of the ball and the equation of motion is $\ddot{x} = -kv - g$ for some constant $k > 0$.

(i) Show that the greatest height achieved is $H = \dfrac{U}{k} - \dfrac{g}{k^2}\log_e\left(\dfrac{g+kU}{g}\right)$. **2**

The ball reaches its maximum height and begins to fall back towards the ground.

Assume that at time t seconds after the ball starts to fall, the displacement is y metres and the velocity is $w\,\text{ms}^{-1}$, from the point at which it begins to fall. Take downward motion as positive. The equation of motion is $\ddot{y} = -kw + g$.

(ii) It hits the ground when $y = H$. Show that in terms of the velocity $W\,\text{ms}^{-1}$ at which the ball hits the ground, **2**

$$H = \frac{-W}{k} + \frac{g}{k^2}\log_e\left(\frac{g}{g-kW}\right).$$

(iii) Let $T = \dfrac{g}{k}$ be the terminal velocity and $U_T = \dfrac{U}{T}$ and $W_T = \dfrac{W}{T}$ be the ratios of the launch and impact speeds to the terminal velocity respectively. Show that **2**

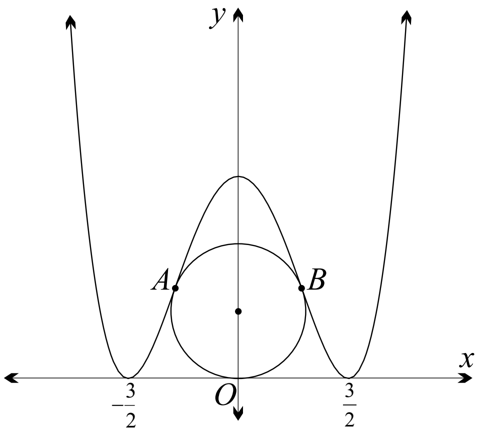
$$U_T + W_T = \log_e\left(\frac{1+U_T}{1-W_T}\right).$$

(iv) Show by substitution that if the ball is thrown at 50% of the terminal velocity then it will impact at approximately 37% of the terminal velocity. **1**

Exam continues on the next page

QUESTION FOURTEEN (Continued)

(c) The circle $x^2 + (y-r)^2 = r^2$ is tangent to the curve $y = \left(x^2 - \frac{3}{2}\right)^2$ at two points A and B, as shown.



(i) Show that at the points of contact, $u^4 - 2u^2 r + u + \frac{3}{2} = 0$, where $u = x^2 - \frac{3}{2}$. **1**

(ii) Explain why $4u^3 - 4ur + 1 = 0$. **1**

(iii) There is a single real solution for u, for which $|x| < \frac{3}{2}$. Find this solution. **2**

(iv) Find the radius of the circle. **1**

End of Section II

END OF EXAMINATION

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ○ B ○ C ○ D ○

Question Two

A ○ B ○ C ○ D ○

Question Three

A ○ B ○ C ○ D ○

Question Four

A ○ B ○ C ○ D ○

Question Five

A ○ B ○ C ○ D ○

Question Six

A ○ B ○ C ○ D ○

Question Seven

A ○ B ○ C ○ D ○

Question Eight

A ○ B ○ C ○ D ○

Question Nine

A ○ B ○ C ○ D ○

Question Ten

A ○ B ○ C ○ D ○

FORM VI EXTENSION 2 2016 MAY ASSESSMENT

SOLUTIONS

(1) (B)

(2) (C)

(3) $2(\cos\pi - i\sin\pi)$

$= 2(1 - 0i)$

$= 2$

$\therefore$ (A) rational

(4) For a polynomial with odd degree, if the leading coefficient, a is positive $a > 0$, as $x \to -\infty$, $y \to -\infty$

and as $x \to \infty$, $y \to \infty$

and therefore (given it is continuous) it will cut the x-axis for some $x \in \mathbb{R}$

or $a < 0$, as $x \to -\infty$ $y \to \infty$

and as $x \to \infty$, $y \to -\infty$

and similarly ...

$\therefore$ (A) is correct.

(B) wrong (since A is correct)

(C) Certainly it's possible to have zero, nonreal zeroes. e.g. $P(x) = (x-2)^3$

(D) ~~Rea~~ For example: $P'(x) = x^2 + 3$ has no real roots

$P(x) = \frac{x^3}{3} + 3x + 2$

$\therefore$ D is wrong.

5

Let $P(x) = x^4 + 2x^3 + 8x + 16$

$\therefore P'(x) = 4x^3 + 6x^2 + 8$

$= 2(2x^3 + 3x^2 + 4)$

$P'(2) > 0$

$P'(-2) = 2(-16 + 12 + 4) = 0$

now $P(-2) = 16 - 16 - 16 + 16 = 0$

$\therefore -2$ is a double root. $\therefore$ C

A: $P(1-\sqrt{3}i) = 0$, but $P'(1-\sqrt{3}i) = -36 - 12\sqrt{3}i \neq 0$

B: $P(1+\sqrt{3}i) = 0$, but $P'(1+\sqrt{3}i) = -36 + 12\sqrt{3}i \neq 0$

D: $P(2) > 0 \neq 0$.

6

$$I = \int_{-2}^{4} x^3\sqrt{16-x^2}\,dx$$

if $f(x) = x^3\sqrt{16-x^2}$

$f(-x) = -x^3\sqrt{16-x^2}$

$= -f(x)$

$$= \int_{-2}^{2} x^3\sqrt{16-x^2}\,dx + \int_{2}^{4} x^3\sqrt{16-x^2}\,dx$$

$$= 0 + \int_{2}^{4} x^3\sqrt{16-x^2}\,dx \qquad \therefore A$$

B would be true if f were an even fn.

(7) since $P(x)$ has real coefficients, its complex zeros must come in conjugate pairs.

$P(3i) = 0 \quad \therefore P(-3i) = 0$

$\therefore (x-3i)(x+3i)$ is a factor

$\therefore (x^2+9)$ is a factor, which is quadratic with no real roots

$\therefore$ (A)

(8) Easiest method! $\dfrac{PS}{PQ} = e$ (Q is closest point on directrix to P)

$$\frac{\frac{16}{5}}{PQ} = \frac{3}{5}$$

$$\therefore PQ = \frac{16}{5} \times \frac{5}{3} = \frac{16}{3}$$

$\therefore$ Q will be $\left(3+\frac{16}{3}, \frac{16}{5}\right)$

$\therefore$ directrix is $x = \frac{25}{3}$. (C)

(9) (A) Certainly possible

(B) acceleration $= 0$ is possible

(C) certainly possible, since a circle of radius 4 makes constant displacement

$\boxed{\text{(D) Not possible}}$

(10) $\overrightarrow{OZ} = z$, $\overrightarrow{ZW} = iz$

$\overrightarrow{OW} = z + iz$

$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OW}$ (by symmetry of triangle OZW)

$\therefore P = \frac{1}{2}(z+iz)$ ~~(D)~~ (C).

(11) (a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^3}}\,dx = -\frac{1}{3}\int (-3x^2)(1-x^3)^{-\frac{1}{2}}\,dx$ (1)

or substⁿ $u = 1-x^3$ or similar

$$= -\frac{1}{3}(1-x^3)^{\frac{1}{2}} \times 2 + c \quad (2)$$

$$= -\frac{2}{3}\sqrt{1-x^3} + c$$

Progress: (1)

Answer: (2)

(b) $P(x) = x^3 - 5x^2 + 11x - 15$

$P(1) = 1 - 5 + 11 - 15 \neq 0$

$P(-1) = -1 - 5 - 11 - 15 \neq 0$

$P(3) = 27 - 45 + 33 - 15 = 0$ (1)

$\therefore x - 3$ is a factor

$$\begin{array}{r} x^2 - 2x + 5 \\ x-3 \,\big)\, x^3 - 5x^2 + 11x - 15 \\ x^3 - 3x^2 \\ \hline -2x^2 \\ -2x^2 + 6x \\ \hline 5x \\ 5x - 15 \\ \hline 0 \end{array}$$

$P(x) = (x-3)(x^2 - 2x + 5)$ (2)

Now, $x^2 - 2x + 5 = (x-1)^2 + 2^2$

$= (x - 1 + 2i)(x - 1 - 2i)$

$\therefore P(x) = (x-3)(x - 1 + 2i)(x - 1 - 2i)$ (3)

(c) (i) ~~$(x - (1-i))(x - (1+i)) = x$~~

$(x - 1 + i)(x - 1 - i) = (x-1)^2 - i^2 = x^2 - 2x + 1 + 1$

$= x^2 - 2x + 2$ (1)

(ii) given (i) $1+i$ and $1-i$ are ~~factors of~~ zeros of $x^2 - 2x + 2$

also $P(x) = x^3 + kx - 5 = (x^2 - 2x + 2)Q(x) + 9x - 9$

11 (c)(ii) $P(x) = x^3 + kx - 5 = (x^2 - 2x + 2)\,Q(x) + 9x - 9$ 1

$P(1+i) = (1+3i-3-i) + k(1+i) - 5 = 0 \times Q(x) + 9 + 9i - 9$

Since $k \in \mathbb{R}$ $\quad -7 + k = 0$ $\quad$ and $\quad i(2+k) = 9i$

$\therefore k = 7$ $\qquad \therefore \underline{k = 7}$. 2

$\underline{\underline{OR}}$ $P(1-i) = (1-3i-3+i) + k(1-i) - 5 = 0 \times Q(x) + 9 - 9i - 9$

$-7 + k + (2-k)i \quad = -9i$

$\underline{k = 7}$ and $\underline{k = 7}$

By Inspection, $A = 2$

(d) (i) $\dfrac{4x^2 + 11}{(2x+3)(x+4)} = \dfrac{2(2x^2 + 11x + 12)}{2x^2 + 11x + 12} + \dfrac{-22x - 13}{(2x+3)(x+4)}$

$\therefore \quad -22x - 13 = B(x+4) + C(2x+3)$ 1

$B + 2C = -22 \quad \cdots ①$

$4B + 3C = -13 \quad \cdots ②$

$4 \times ① - ②$ $\quad 0B + 5C = -88 + 13 = -75$

$\underline{C = -15}$

sub in ① $\quad B - 30 = -22$

$\underline{B = 8}$ 2

(ii) $\displaystyle\int_{-1}^{1} \frac{4x^2 + 11}{(2x+3)(x+4)}\,dx = \int_{-1}^{1} \left(2 + \frac{8}{2x+3} - \frac{15}{x+4}\right) dx$

$= \Big[2x\Big]_{-1}^{1} + 4\Big[\ln(2x+3)\Big]_{-1}^{1} - 15\Big[\ln(x+4)\Big]_{-1}^{1}$ 1

$= 4 + 4\ln 5 - 4\ln 1 - 15\ln 5 + 15\ln 3$

$= \underline{4 + 15\ln 3 - 11\ln 5}$ 2

11 (e)

$$F = 18 - 8x = m\ddot{x} = 4\ddot{x}$$

$$\ddot{x} = \frac{9}{2} - 2x$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{9}{2} - 2x \quad ①$$

$$\frac{1}{2}v^2 = \frac{9}{2}x - x^2 + c_1$$

$$v^2 = 9x - 2x^2 + c_2 \quad ②$$

$x = 4, v = 2$: $4 = 36 - 32 + c_2 \Rightarrow c_2 = 0$ must be evaluated for ③ marks to be awarded.

$$\therefore v^2 = 9x - 2x^2 \quad ③$$

12 (a) $x = 3\sec\theta$, $y = 2\tan\theta$

(i) $\sec^2\theta = 1 + \tan^2\theta$

$$\left(\frac{x}{3}\right)^2 = 1 + \left(\frac{y}{2}\right)^2$$

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \quad 1$$

(ii) asymptotes at $y = \pm\frac{2}{3}x$ 1

(iii) $b^2 = a^2(e^2-1)$

$$\frac{4}{9} = e^2 - 1$$

$$e^2 = \frac{13}{9}$$

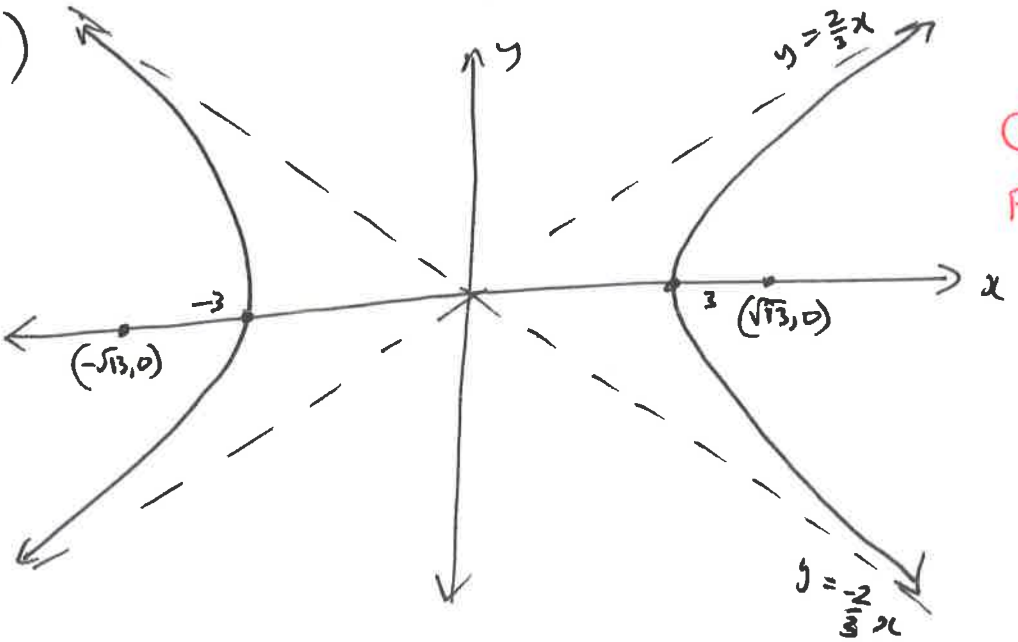
$$e = \frac{\sqrt{13}}{3} \quad 1$$

Focus $(ae, 0)$ or $(-ae, 0)$

$= \left(\frac{\sqrt{13}}{3}\times 3, 0\right)$ or $\left(-\frac{\sqrt{13}}{3}\times 3, 0\right)$

$= (\sqrt{13}, 0)$ or $(-\sqrt{13}, 0)$ 2

(iv)



2 marks
Feature omitted or incorrect according to student's working -1.

(b) $\ddot{x} = -g + kv^2$ at $\ddot{x} = 0$, $v = V_T$ 1

$0 = -g + kV_T^2$

$V_T^2 = \frac{g}{k}$

$V_T = \sqrt{\frac{g}{k}}$ 2

This solution was to a previous draft of the question. The corrected solution follows.

Corrected solution for Q12 (b):

(b)

$$m\ddot{x} = +mg - kv^2$$

$$\ddot{x} = g - \frac{k}{m}v^2 \quad (1) \quad \text{terminal velocity when } \underline{\ddot{x} = 0}$$

$$0 = g - \frac{k}{m}V_T^2$$

$$V_T^2 = \frac{gm}{k}$$

$$V_T = \sqrt{\frac{gm}{k}} \quad (2) \quad (\text{since down is positive, } V_T > 0)$$

12 (c) $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = 0, \ \theta = 0$

$x = 1, \ \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$$\int_0^1 \frac{1}{(4-x^2)^{3/2}}\, dx = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{(4(1-\sin^2\theta))^{\frac{3}{2}}} \quad 1$$

$$= \frac{1}{4}\int_0^{\pi/6} \frac{\cos\theta}{\cos^3\theta}\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\left(\tan\theta\right)_0^{\frac{\pi}{6}} \quad 2$$

$$= \frac{1}{4} \times \frac{1}{\sqrt{3}} = \frac{1}{4\sqrt{3}} \quad 3$$

12 (d) $P(x) = \cancel{5x}\ x^3 - 5x^2 - 2x - 8$

(i) $P(2x)$ will have zeroes $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ 1

$P(2x) = 8x^3 - 20x^2 - 4x - 8$

$Q(x) = 2x^3 - 5x^2 - x - 2$ (simplified) 2

(ii) $\frac{3\alpha}{2} + \beta + \gamma, \ \alpha + \frac{3\beta}{2} + \gamma, \ \alpha + \beta + \frac{3\gamma}{2}$ are equal to:

$(\alpha+\beta+\gamma) + \frac{\alpha}{2}, \ (\alpha+\beta+\gamma) + \frac{\beta}{2}, \ (\alpha+\beta+\gamma) + \frac{\gamma}{2}$ 1

now $\alpha+\beta+\gamma = -(-5) = 5$ (from $P(x)$)

$\therefore$ Required zeroes ~~roots~~ are $\frac{\alpha}{2}+5, \ \frac{\beta}{2}+5, \ \frac{\gamma}{2}+5$

$\therefore$ $Q(x-5)$ will give these zeroes 2

$$Q(x-5) = 2(x-5)^3 - 5(x-5)^2 - (x-5) - 2$$

$$= 2x^3 - 30x^2 + 150x - 250$$
$$- 5x^2 + 50x - 125$$
$$- x + 5$$
$$- 2$$

$$\therefore T(x) = 2x^2 - 35x^2 + 199x - 372 \quad 3$$

(13a) $\dfrac{x^2}{225} - \dfrac{y^2}{144} = 1$ $a = 15$ $b = 12$ $216x - 135y = 3240$

(i) Chord of contact @ $(15, 6)$

$\dfrac{15x}{225} - \dfrac{6y}{144} = 1$ $\dfrac{x}{15} - \dfrac{y}{24} = 1$

OR $y = \frac{8}{5}(x - 15)$

$y = \frac{8x}{5} - 24$

OR $8x - 5y - 120 = 0$

(ii) Solve Chord of Contact & Hyp Simultaneously

$y = \frac{8}{5}(x - 15) = \frac{8}{5}(x - 24)$

$\therefore \dfrac{x^2}{15^2} - \dfrac{8^2(x-15)^2}{5^2 \times 12^2} = 1$ ✓

$(\times 15^2)$

$x^2 - 4(x-15)^2 = 225$

$x^2 - 4(x^2 - 30x + 225) = 225$

$-3x^2 + 120x - 900 = 225$

$3x^2 - 120x + \underset{5 \times 225}{} = 0$

$x^2 - 40x + 5^2 \times 15 = 0$

$(x - 25)(x - 15) = 0$ ✓

$x = 15$ or $x = 25$

$y = 0$ $y = 16$.

$x = \frac{5}{8}(y + 24)$

$x = \frac{5}{8}y + 15 = \dfrac{120 + 5y}{8}$

$\dfrac{x^2}{15^2} = \left(1 + \dfrac{y}{24}\right)^2$

$\left(1 + \dfrac{y}{24}\right)^2 - \dfrac{y^2}{12^2} = 1$

$\dfrac{1}{24^2}(24 + y)^2 - \dfrac{4y^2}{24^2} = 1$

$24^2 + 48y + y^2 - 4y^2 = 24^2$

$48y - 3y^2 = 0$

$3y(16 - y) = 0$

ie $\dfrac{15x}{225} = 1$

$x = 15$

Vertical

or

$\dfrac{25x}{225} - \dfrac{16y}{144} = 1$

$\dfrac{x}{9} - \dfrac{y}{9} = 1$

$x - y - 9 = 0$ ✓

or $y = x - 9$

(13) (b) $I_n = \int_0^1 (1+x^2)^n \, dx$

$dv = dx$
$v = x$

$u = (1+x^2)^n$

$du = n(1+x^2)^{n-1} \times 2x \, dx$

$$\therefore I_n = \left[(1+x^2)^n x\right]_0^1 - \int_0^1 x \cdot n(1+x^2)^{n-1} \cdot 2x \, dx \quad (1)$$

$$= (2^n - 0) - 2n \int_0^1 x^2 (1+x^2)^{n-1} dx$$

NB $\boxed{x^2 = (1+x^2) - (1)}$

$$= 2^n - 2n\left[\int_0^1 (1+x^2)(1+x^2)^{n-1} dx - \int_0^1 (1+x^2)^{n-1} dx\right] \quad (2)$$

$$= 2^n - 2n(I_n - I_{n-1})$$

$$I_n(1+2n) = 2^n + 2n I_{n-1} \quad (3)$$

$$I_n = \frac{2^n}{2n+1} + \frac{2n}{2n+1} I_{n-1}$$

c) $P(x) = x^3 - 3x^2 - 2x - 1 = 0$ roots α, β & γ

(i) Replace x with $\sqrt{x}$

$$P(\sqrt{x}) = x\sqrt{x} - 3x - 2\sqrt{x} - 1 = 0 \checkmark$$

$$\therefore \sqrt{x}(x-2) = 3x+1$$

$$x(x^2 - 4x + 4) = 9x^2 + 6x + 1 \checkmark$$

$$\underline{x^3 - 13x^2 - 2x - 1 = 0} \checkmark$$

(ii) $\sum \alpha\beta = \frac{c}{a} = (-2)$ $\quad \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = \underline{(-2)}$ $\checkmark$

d) (i) Tangent $x + t^2 y = 2ct$

at P $y = 0$ $\therefore$ $x = 2ct$ $\quad$ $\underline{P(2ct, 0)}$

at Q $x = 0$ $\therefore$ $t^2 y = 2ct$

$$y = \frac{2c}{t} \quad \underline{Q\left(0, \frac{2c}{t}\right)} \checkmark$$

(ii) Normal $t^2 x - y = ct^3 - \frac{c}{t}$ (1) intersects hyperbola $xy = c^2$ (2) at T and V

from (2) $y = \frac{c^2}{x}$ $\therefore$ $t^2 x - \frac{c^2}{x} = ct^3 - \frac{c}{t}$

$(\times x)$ $(\div t^2)$ $\quad x^2 - \frac{c^2}{t^2} = xct - \frac{cx}{t^3}$

$x^2+\left(\frac{c}{t^3}-ct\right)x-\frac{c^2}{t^2}=0$ ✓ or $x^2t^2+x\left(\frac{c}{t}-ct^3\right)-c^2=0$

$\cancel{x^2+\frac{c(1-t^4)}{t^3}x-\frac{c^2}{t^2}=0}$

$x^2-ctx+\frac{cx}{t^3}-\frac{c^2}{t^2}=0$

$x(x-ct)+\frac{c}{t^3}(x-ct)=0$

$(x-ct)\left(x+\frac{c}{t^3}\right)=0$

$\therefore x=ct$ at pt T or $x=-\frac{c}{t^3}$ ie at V

sub into ② $\quad -\frac{c}{t^3}\times y=c^2$

$y=\frac{c^2t^3}{-c}$

$=-ct^3$ ✓

$V\left(-\frac{c}{t^3},-ct^3\right)$ as required

$\boxed{\text{(ii) (Alternative)}}$ $xy=c^2$.

$x=\frac{c^2}{y}$

$t^2x-y=ct^3-\frac{c}{t}$

$\frac{c^2t^2}{y}-y=ct^3-\frac{c}{t}$

$c^2t^3-ty^2=ct^4y-cy$.

$ty^2+(ct^4-c)y-c^2t^3=0$

$y^2+\frac{c}{t}(t^4-1)y-c^2t^2=0$

$y\left(y-\frac{c}{t}\right)+ct^3\left(y-\frac{c}{t}\right)=0$

$(y+ct^3)\left(y-\frac{c}{t}\right)=0$

(iii) $P(2ct, 0)$ $Q\left(0, \frac{2c}{t}\right)$ $V\left(-\frac{c}{t^3}, -ct^3\right)$

$$PV^2 = \left(2ct + \frac{c}{t^3}\right)^2 + \left(0 + ct^3\right)^2$$

$$= 4c^2t^2 + \frac{4c^2}{t^2} + \frac{c^2}{t^6} + c^2t^6$$

$$= c^2\left(t^6 + 4t^2 + \frac{4}{t^2} + \frac{1}{t^6}\right) \text{ or } \frac{c^2}{t^6}\left(t^{12} + 4t^8 + 4t^4 + 1\right)$$

$$QV^2 = \left(0 + \frac{c}{t^3}\right)^2 + \left(\frac{2c}{t} + ct^3\right)^2$$

$$= \frac{c^2}{t^6} + \frac{4c^2}{t^2} + 4c^2t^2 + c^2t^6$$

$$= c^2\left(t^6 + 4t^2 + \frac{4}{t^2} + \frac{1}{t^6}\right)$$

$\therefore PV^2 = QV^2$

$PV, QV > 0$ $\therefore$ $PV = QV$ hence $\triangle PQV$ is isosceles.

<u>OR</u> (easier)

<u>T is midpoint of PQ</u>: $\left(\frac{2ct+0}{2}, \frac{0+\frac{2c}{t}}{2}\right)$

$= \left(ct, \frac{c}{t}\right)$ (1)

$\therefore$ $TQ = TP$

TV is common

$\angle QTV = \angle PTV = 90°$

$\therefore \triangle QTV \equiv \triangle PTV$ (SAS) (2)

$\therefore QV = PV$ (matching sides in congruent $\triangle$'s)

14 (a) $x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$\therefore \frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ 1

$\therefore$ gradient of normal is $\frac{a\sin\theta}{b\cos\theta}$

$\therefore$ Eqn normal: $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$ 2

$\times \frac{b}{\sin\theta}$ $\quad \frac{by}{\sin\theta} - b^2 = \frac{ax}{\cos\theta} - a^2$

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ 3

14 (b) (i) $\ddot{x} = -kv - g$

$v \cdot \frac{dv}{dx} = -(g + kv)$

$\frac{dx}{dv} = -\frac{v}{g+kv} = -\frac{1}{k}\left(\frac{g+kv}{g+kv}\right) + \frac{1}{k}\frac{g}{g+kv}$

$= -\frac{1}{k} + \frac{g}{\cancel{k}}\left(\frac{k}{g+kv}\right)$

$\therefore x = \frac{-v}{k} + \frac{g}{k^2}\ln(g+kv) + c_1$ 1

now, when $x=0$, $v=U$ $\quad \therefore c_1 = \frac{U}{k} - \frac{g}{k^2}\ln(g+kU)$

$x = \frac{U-v}{k} + \frac{g}{k^2}\ln\left(\frac{g+kv}{g+kU}\right)$

now when $v=0$, $x=H$:

2

$H = \frac{U}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g+kU}\right) = \frac{U}{k} - \frac{g}{k^2}\ln\left(\frac{g+kU}{g}\right)$

14 (b)

(ii) $\ddot{y} = -kw + g$

$$w \cdot \frac{dw}{dy} = g - kw$$

$$\frac{dy}{dw} = \frac{w}{g-kw} = -\frac{1}{k}\,\frac{g-kw}{g-kw} + \frac{g}{k}\cdot\frac{1}{g-kw}$$

$$= -\frac{1}{k} - \frac{g}{k^2}\,\frac{-k}{g-kw}$$

$$y = -\frac{w}{k} - \frac{g}{k^2}\ln(g-kw) + c_2 \quad 1$$

when $y=0, w=0$ $\therefore$ $c_2 = \frac{g}{k^2}\ln g$

$$\therefore y = -\frac{w}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kw}\right)$$

now when $y = H$, $w = W$ 2

$$\therefore H = -\frac{W}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kW}\right)$$

(iii) equating H expressions:

$$\frac{U}{k} - \frac{g}{k^2}\ln\left(\frac{g+kU}{g}\right) = -\frac{W}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kW}\right)$$

$\times k$

$$U + W = \frac{g}{k}\ln\left(\frac{g+kU}{g}\cdot\frac{g}{g-kW}\right)$$

$\div T$ or $\times \frac{k}{g}$

$$\frac{U}{T} + \frac{W}{T} = \ln\left(\frac{g+kU}{g-kW}\right) \quad 1$$

$$U_T + W_T = \ln\left(\frac{1+\frac{U}{g/k}}{1-\frac{W}{g/k}}\right)$$

$$U_T + W_T = \ln\left(\frac{1+U_T}{1-W_T}\right) \quad 2$$

14 (b) (iv)

$u_T = 50\%$ $w_T = 37\%$ gives:

$LHS = 87\%$

1

$RHS = \ln\left(\frac{1.5}{0.63}\right) \approx 86.8\%$

$\approx LHS$.

14 (c) $x^2+(y-r)^2=r^2 \qquad y=\left(x^2-\frac{3}{2}\right)^2$

(i) if $u=x^2-\frac{3}{2}$

$x^2=u+\frac{3}{2}$ and $y=u^2$

so $u+\frac{3}{2}+(u^2-r)^2=r^2$

$u+\frac{3}{2}+u^4-2u^2r+r^2=r^2$

$u^4-2u^2r+u+\frac{3}{2}=0$ 1

(ii) Now, since the curves are tangent at these points, the eqn in u must have a double root, so the derivative must also ~~be~~ give the same values for u. 1

$4u^3-4ur+1=0$

(iii) equating r's gives:

$$\frac{4u^3+1}{4u}=\frac{u^4+u+\frac{3}{2}}{2u^2}$$ 1

$\times 4u^2$

$4u^4+u=2u^4+2u+3$

$2u^4-u-3=0$

now $u=-1$ is a solution $2(1)-(-1)-3=0$

2

(iv) $u=-1$, $r=\frac{4\times(-1)+1}{4(-1)}=\frac{3}{4}$. 1

also $r=\frac{1-1+\frac{3}{2}}{2}=\frac{3}{4}$.

(Also $x=\pm\frac{1}{\sqrt{2}}$, $y=1$)

(Another real solution exists, $u\approx 1.204$, $r\approx 1.657$, $(x,y)\approx(\pm 1.644, 1.450)$ but it is outside the given domain.)